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Neutrino radiation fields in general relativity

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Abstract. It is shown that there exist Einstein–neutrino fields which are not defined uniquely by the metric of space–time. These fields form a subclass of the neutrino pure radiation fields and are analogous to Peres' exceptional case for Einstein–Maxwell null fields.

1. On the uniqueness of the neutrino field

In the theory of geometrodynamics it has been found possible to describe gravitational and electromagnetic fields purely in terms of metric concepts. In fact, if the metric tensor of a space–time describing such fields is known then the field is determined exactly, apart from a constant duality rotation—except for Peres' exceptional case (Peres 1961, Geroch 1966)—that is, for twist-free null electromagnetic fields.

The analogous case of combined gravitational and neutrino fields has not been so thoroughly investigated. However, we can state the following theorem, which has been partially arrived at by Bergmann (1960).

Theorem 1. If a space–time admits a neutrino field and the neutrino flux vector is prescribed, then the neutrino spinor field is uniquely determined up to a constant phase factor by the metric of space–time.

In general, however, the neutrino flux vector will not be prescribed and there will exist space–times in which the neutrino field is not uniquely determined. This situation is best described by the following theorems.

Theorem 2. If a given space–time admits an Einstein–neutrino field with positive energy density and flux vector l_μ, and also a second field with flux vector $e^{2\psi}l_\mu$, then the fields are both pure radiation fields.

Theorem 3. If a given space–time admits a pure-radiation Einstein–neutrino field with flux vector $l_\mu = \lambda U,_\mu$, where λ and U are scalar functions, and if the Ricci tensor of the space–time is of the form

$$R_{\mu\nu} = \lambda F(U) U,_\mu U,_\nu$$

then the space–time admits a multiplicity of neutrino radiation fields with flux vectors $l_\mu = \lambda f(U) U,_\mu$ where $f(U)$ is an arbitrary function of U. The neutrino field is then not determined uniquely by the metric.

An example of this case is the exact solution given by Griffiths and Newing (1970) in which $\lambda = 1$ and coordinates are chosen so that $U = x^1$. A more complete examination of the uniqueness of the neutrino field will be considered in a later paper.

2. Proof of the theorems

Let the spinor ξ_A define the neutrino field. Then if the flux vector l_μ (for notation see Griffiths and Newing 1970) is prescribed, the only other spinors which could describe the neutrino field have the form

$$\xi_A{}' = e^{i\eta}\xi_A.$$

Now the energy–momentum tensor for the neutrino field is

$$E_{\mu\nu} = 2i(\sigma_{(\mu}{}^{A\dot{B}}\xi_{A|\nu)}\xi_{\dot{B}} - \sigma_{(\mu}{}^{A\dot{B}}\xi_{\dot{B}|\nu)}\xi_A)$$

and so

$$E_{\mu\nu}{}' = E_{\mu\nu} - 2(l_\mu\eta,_\nu + \eta,_\mu l_\nu).$$

Thus since the Ricci tensor of the space–time, and hence $E_{\mu\nu}$, is known, $\xi_A{}'$ can only describe a neutrino field if η is a constant. This establishes theorem 1.

Suppose that in a given space–time l_μ and $e^{2\psi}l_\mu$ both define neutrino fields. If the spinor ξ_A defines the first field then $e^{\phi}\xi_A$, where $\phi = \psi + i\eta$, defines the second. Since both spinor fields satisfy the neutrino equation

$$\xi^A{}_{|A\dot{B}} = 0 \qquad (e^{\phi}\xi^A)_{|A\dot{B}} = 0$$

we require that

$$\phi_{|A\dot{B}} = \xi_A(a\xi_{\dot{B}} + b\chi_{\dot{B}}).$$

In terms of the tetrad (defined in Griffiths and Newing 1970) this can be represented as

$$\phi,_\mu = al_\mu + b\bar{m}_\mu. \tag{2.1}$$

The energy–momentum tensors of the two fields are given by

$$\underset{(1)}{E_{\mu\nu}} = 2i(\sigma_{(\mu}{}^{A\dot{B}}\xi_{A|\nu)}\xi_{\dot{B}} - \sigma_{(\mu}{}^{A\dot{B}}\xi_{\dot{B}|\nu)}\xi_A)$$

and

$$\underset{(2)}{E_{\mu\nu}} = 2ie^{2\psi}(\sigma_{(\mu}{}^{A\dot{B}}\xi_{A|\nu)}\xi_{\dot{B}} - \sigma_{(\mu}{}^{A\dot{B}}\xi_{\dot{B}|\nu)}\xi_A + l_{(\mu}\phi,_{\nu)} - l_{(\mu}\bar{\phi},_{\nu)}).$$

Thus

$$\underset{(2)}{E_{\mu\nu}} = e^{2\psi}\{\underset{(1)}{E_{\mu\nu}} + 2i(a-\bar{a})l_\mu l_\nu + 2ibl_{(\mu}\bar{m}_{\nu)} - 2i\bar{b}l_{(\mu}m_{\nu)}\}.$$

But since the two fields have the same energy–momentum tensors we can put

$$(1-e^{-2\psi})E_{\mu\nu} = -2i(a-\bar{a})l_\mu l_\nu - 2ibl_{(\mu}\bar{m}_{\nu)} + 2i\bar{b}l_{(\mu}m_{\nu)}.$$

If the field has positive energy density then we must have $b = 0$. This has been shown by Griffiths and Newing (1971) and independently by Wainwright (to be published). Both fields therefore are pure radiation fields. This completes the proof of theorem 2.

Suppose now that we have a radiation field with flux vector $l_{(1)\mu} = \lambda U,_\mu$ and that the energy–momentum tensor is

$$E_{\mu\nu} = -\lambda F(U)U,_\mu U,_\nu$$

where $F(U)$ is some given function of U. The space–time will admit a second field with flux vector $l_{(2)\mu} = \lambda e^{2\psi}U,_\mu$ if

$$F(U) = \frac{2i\lambda(a-\bar{a})}{(1-e^{-2\psi})} \tag{2.2}$$

and (2.1) is satisfied with $b = 0$. Now put $a = x + iy$, then

$$\psi,_\alpha = xl_\alpha = x\lambda U,_\alpha \qquad \eta,_\alpha = yl_\alpha = y\lambda U,_\alpha$$

and hence ψ, η, $x\lambda$ and $y\lambda$ must all be functions of U. From (2.2) we have

$$y = -\frac{1}{4\lambda}(1 - e^{-2\psi})F(U)$$

and this equation may be taken to define y for any arbitrary function ψ of U. Theorem 3 then follows.

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